THE MATHEMATICS OF ISOCHRONISM IN GALILEO: FROM HIS MANUSCRIPT NOTES ON MOTION TO THE *DISCORSI*[1]

Mohammed ABATTOUY*

Abstract. This article surveys Galileo's contribution to the conceptualisation of the problem of isochronism, as his most important input in the study of the oscillation of heavy bodies. We will deal essentially with the mathematical aspects of his various attempts to discover mathematical proofs for his theorems of isochronism, in his early manuscripts (1600-1609), bound in Volume 72 of the Galilean Manuscripts, and in his later two major publications, the *Dialogo* and the *Discorsi*. The experimental procedures he implemented in his research will be dealt with as far as they shed light on different aspects of properly mathematical issues of the Galilean analysis of iscohronism. Alternating between the analysis of Galileo's private manuscripts and the relevant passages in his mature works, we follow the emergence and evolution of Galileo's theory of the pendulum within his physics of motion, and we witness how his investigation was faced with serious challenges, mathematical and experimental, that he could never overcome completely.

Keywords: Isochronism, Galileo Galilei, theory of the pendulum, law of isochronism, law of length, experiments with pendulum, modern physics, chronometry.

1. Introduction

Two major subjects dominated physics in the 17th century, those of acceleration and oscillation of bodies. The works of Galileo in physics provide a perfect illustration of this statement. Almost all his achievements in the study of motion were dedicated precisely to these two themes. Less known than his researches on acceleration, his work on oscillation, especially on isochronism, is no less important on the mathematical and methodological levels. Very early, at the beginning of the century, as we learn from his correspondence and from his manuscript papers, we find him struggling with the question of isochronism, that he tried to establish on a rigorous mathematical foundation, before dealing with that of acceleration. It was while studying the latter that he discovered the law of free fall that he demonstrated, in an early attempt, on the basis of a property of isochronism.

* Mohamed V University in Rabat, Faculty of Letters and Humanities, Department of Philosophy, Avenue Ibn Battouta, PO Box 1040, Rabat, Morocco. E-mail: abattouy.mohammed@gmail.com.

This article aims to survey Galileo's contribution to the conceptualisation of the problem of isochronism, as his most important contribution to resolving the problem of the oscillation of heavy bodies. We will deal essentially with the mathematical aspects of his various arduous attempts to establish his theorems of isochronism on a solid mathematical foundation, in his early manuscripts (1600-1609) and in his later two major publications, the *Dialogo sopra i due massimi sistemi* (1632) and the *Discorsi e dimostrazioni matematiche* (1638). The experimental procedures he implemented in his research will be addressed only as far as they would shed light on the properly mathematical issues.

Until recently, few general studies were devoted to the question of isochronism in Galileo. This situation changed drastically as several publications were released and yielded important results.[2] In parallel, Galilean studies have been enriched in recent decades by a huge quantity of articles and books on Galileo's personal notes recording his early investigations on the basic properties of motion. These studies painted a rich and informative picture of the genesis of Galileo's "new science of motion," in which several documents are directly related to isochronism. Even though we know more, at present, about his theory of pendulum and isochronism, the connections between Galileo's work on the pendulum theory and the mathematical propositions of isochronism deserve to be reexamined, with the aim of analysing this link thoroughly. In the following essay, I propose to provide a survey of this important topic, by alternating between the analysis of Galileo's private manuscripts and the relevant passages in his mature works. Thus we follow the emergence and evolution of Galileo's theory of the pendulum within his physics of motion, and we witness how the great physicist confronted serious challenges, mathematical and experimental, that he could never overcome completely.

2. Setting the stage

In the Galilean science of motion, the problem of isochronism refers to the physical situation when the equality of times for motions of one or several bodies is obtained. More precisely, it concerns determining the conditions in which periods of descent on internal cords (considered as many inclined planes) of a vertical circle or during oscillations along arcs of a circle are constant. The first case describes what may be called the isochronism of cords, whereas the second concerns the pendulum isochronism. In his later main publications, the *Dialogo* and especially the *Discorsi*, Galileo addressed the question of isochronism according to those two complementary angles, for cords and for arcs of a circle. The first case was stated and proved as the sixth theorem of accelerated motion; it is known as 'Galileo's Theorem' or the law of cords:

> Theorem VI. If from the highest or lowest point in a vertical circle there be drawn any inclined planes meeting the circumference, the times of descent along these cords are each equal to the other.[3]

The pendulum isochronism will not have this privilege of being stated as a theorem. In the First Day of the *Discorsi*, it is introduced in a discursive way as valid for small, large, and mean oscillations, and then an experiment is reported to provide support to this assertion, describing the bahaviour of two unequal heavy balls, one in cork and the other in lead. The two balls are suspended by two equal threads and begin to oscillate when they are discarded from the vertical in the same time:

> This free vibration repeated a hundred times showed clearly that the heavy body maintains so nearly the period of the light body that neither in a hundred swings nor even in a thousand will the former anticipate the latter by as much as a single moment, so perfectly do they keep step.[4]

Galileo never happened to establish this proposition on a mathematical proof, so he validated it with the law of cords and with various experimental set-ups supposed to provide it with the required confirmation. However, the lack of mathematical proof did not prevent him from considering it as solid enough to support a theory of the pendulum in which the properties of the pendulum were invested, sometimes successfully, for the conception of various devices to measure time.

Among the sources of information on the procedures enforced by Galileo in his investigations on the properties of motion, we have access to a rich manuscript containing various notes and fragments, many of which are autographs written in Galileo's hand. These documents provide a valuable help in reconstructing the evolution of his early investigations on motion. Collected in Volume 72, a codex of the *Manoscritti Galileiani* preserved at the *Biblioteca Nazionale* in Florence, these papers have been a hot spot for Galilean studies for almost five decades.

Volume 72 contains all the manuscript materiel that was preserved from Galileo's original researches on motion from around 1600 until the final writing of the *Discorsi* (1631-1636). Composed of 241 folios, the codex contains texts, mathematical demonstrations, diagrams and calculations. These documents record the first versions of the discoveries from which derive most of the theorems of the *De motu locali*, the mathematical treatise published in the last two parts of the *Discorsi*, respectively on uniform motion (DML-1), accelerated motion (DML-2), and on the motion of projectiles (DML-3). The major part of these papers were written by Galileo in different periods of his research in kinematics of motion, mainly in Padua from 1600 until mid-1609, in Florence between 1616 and 1618, and during the preparation of the final version of the *Discorsi* from 1631 onwards.[5]

The interest of these manuscript materiels was recognized by historians of science in an early stage of Galilean studies. Caverni studied some of them in the late 19th century,[6] then most of the codex was published by Favaro, the editor of the *Edizione Nazionale* of Galileo's works, in volume VIII, as an appendix to the *Discorsi*. But these private papers underwent a real historiographic renaissance only in the 1970s, when they became the centre of interest of historical studies aiming at reconstructing the chronology of Galileo's discoveries on motion. The study of this materiel is not easy, as the documents are for the most part undated fragments bound together without any order, which makes their interpretation an arduous task. Despite these difficulties, modern history of science drew from their analysis valuable conclusions. One of the most important results was setting a relative chronology describing the main stages of the emergence and evolution of the new science of motion presented by Galileo to his readers in the three books of DML in 1638.

Concerning the pendulum theory and the various aspects of isochonism, several documents of Volume 72 show diverse ways in which Galileo tried to construct the mathematical demonstration of the law of isochronism. They depict clearly a case in point of his method and constitute an actual instance of the way he conceived of the entreprise of mathematising physical pheonomena. Using mathematics as his main tool of investigation, he devoted to experiment on the whole a secondary role in this matter. In this particular case, experiment had only a loose agreement with his proposition, and was even at loggerheads with the general assertion of the pendulum isochronism. However, this gap between theory and experiment did not cause Galileo to reject the validity of his theorem. On the contrary, supporting the isochronism of the arcs of circle by the isochronism of cords, he continued to defend the validity of these two aspects of his theory of isochronism.

3. The isochronism of cords

From November 1602 onwards, Galileo claimed that the motions of a simple pendulum were isochronous, although he admitted that he had no firm proof supporting it. This situation continued all along his scientific career, when he published in the *Discorsi* the different arguments of his pendulum theory. This important issue received different historiographical assessments. While some historians speculated that Galileo must have relied on "a wider range of evidence than he indicated in the *Discorsi*,"[7] others claimed that he published the isochronism of the circular pendulum even though he knew it to be false,[8] and that his claim about the isochronism of the pendulum was "based more on mathematical deduction than on experimental observation."[9] More recently, it was shown that light pendulums set to swing from modest angles can indeed be isochronous; however, by using heavier pendulums or greater angles, the isochronism of the simple pendulum breaks down. Galileo could not have failed to notice this phenomenon by himself, which must have certainly confirmed his conviction that experience does not teach the causes of natural processes and, in turn, neutralized the problem of discrepancy from isochrony.[10]

3.1. The discovery announcement

A 1602 letter is the earliest surviving document in which Galileo discusses the hypothesis of pendulum's isochronism. In the letter, Galileo claimed that all pendulums are isochronous. He added that he had long been trying to demonstrate isochronism on the basis of "mechanical arguments," but that so far he had been unable to do so. In brief, as we shall see, from 1602 onwards, Galileo referred to pendulum isochronism as an admirable property but failed to demonstrate it.

On 29 November 1602 Galileo wrote in Padua a letter to Guidobaldo del Monte,[11] his patron and protector, in which he disclosed three propositions that he just discovered: (a) a first proposition stating the *pendulm isochronism* and asserting that large as well as small oscillations of a pendulum occur in equal times for the same length of the thread; (b) a second proposition known in the literature as the *cords* *isochronism* according to which a body moving downwards along any cords drawn from the lowest point or the highest point of a vertical circle and meeting the circumference in any point, will perform all its descents in equal times. This proposition is sometimes referred to as "Galileo's theorem."[12] To these two general statements Galileo added, by extension, a third proposition on the ratios of the times of descent on internal cords of the lowest quarter of a vertical circle: (c) a body requires more time to descend on a cord drawn from the circumference of the circle towards its lowest point than to descend on two combined cords subtending the arc determined by the first cord.[13] The second and third propositions were numbered in the *Discorsi* as Theorems VI and XXII of the accelerated motion.

We understand from Galileo's letter that this missive was part of a larger correspondence with Guidobaldo including previous epistolary exchanges on the same subject. First, Galileo announced the discovery of the cords isochronism. Surprised by this result and unable to provide a conclusive experimental corroboration for it, Guidobaldo expressed his skepticism to Galileo concerning the isochronism of circular motion. Then Galileo answered him in the long missive dated November 29, 1602, in which he adviced to use the pendulum to reconcile the mathematical proposition with real motion:

> I take two thin strings, equally long about two or three *braccia*, let them be AB, EF, and attach them to two nails, A and E, while at the other ends I tie two equal lead balls (although it would make no difference if they were different) [*Figure 1*]. After removing the strings from the vertical, one a lot, as along arc CB, the other very little, as along arc IF, I let them go at the same moment. One begins to describe great arcs, similar to BCD, while the other describes small arcs, similar to FIG; yet mobile B does not employ more time traversing the whole arc BCD, than the other mobile, F, traversing arc FIG.[14]

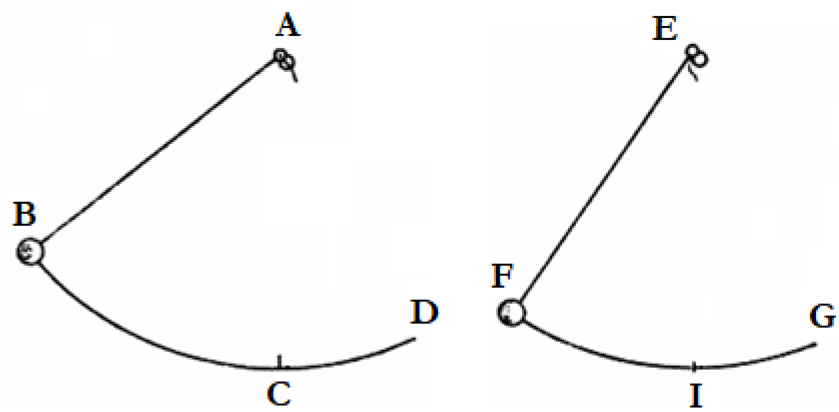


Figure 1

Then he stated the two theorems related respectively to the isochronism of cords and to the properties of the times of descent on the internal cords of the lowest quadrant of a vertical circle:[15]

> Let diameter AB, in circle BDA, be perpendicular to the horizon, and from point A let lines be drawn to the circumference, such as AF, AE, AD, AC [*Figure 2*]: I prove that equal bodies fall in the same time along the vertical BA and the inclined planes CA, DA, EA, FA. Thus, if they start at the same moment from points B, C, D, E, F, they will arrive at the same moment at point A, no matter how small is line FA. The following, which I have also demonstrated, may perhaps appear even more incredible. If the line is not greater than the cord of a quadrant, and if the lines, SI, IA, are taken as one pleases, the same body will more quickly traverse path SIA, starting from S, than the single path IA, starting from I."[16]

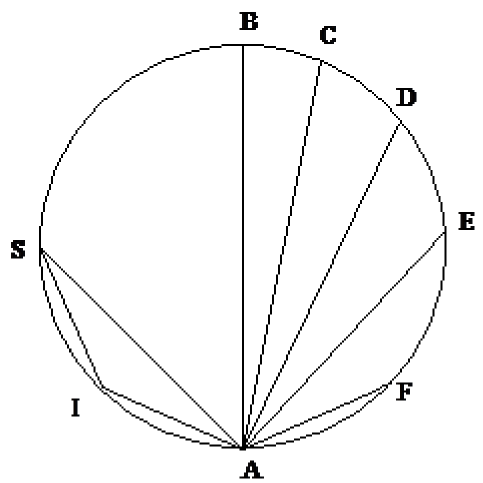


Figure 2

In terms of the diagrams drawn in the letter, the three propositions may be formulated as follows:

– The pendulum isochronism : t(FIG) = t(BCD) [*Figure 1*];
– The cords isochronism : t(FA) = t(EA)... = t(BA) = t(IA) [*Figure 2*];
– Theorem XXII : t(SIA) < t(SA) [*Figure 2*].

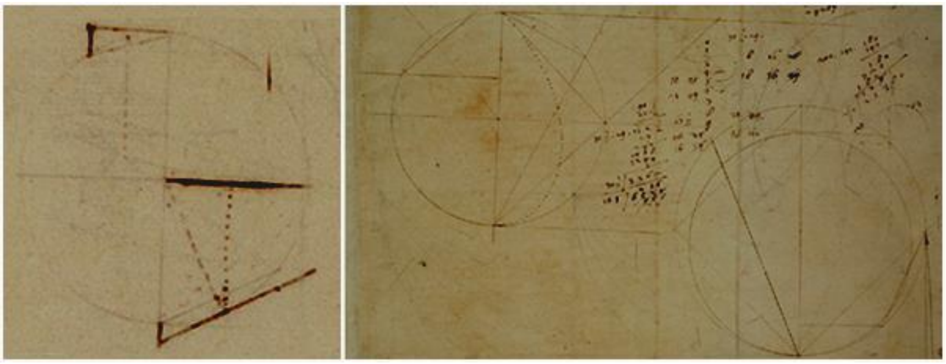
The letter to Guidobaldo is a document of the outmost historical value. In it Galileo announced no less than the first three propositions of the new science of motion and brings a clear testimony as to the success of his early investigations for building general propositions which are compatible with experience. Further, it is a firmly dated document that constitutes thus a valuable temporal landmark to which we can report several of Galileo's early manuscript notes written at the beginning of the century and in which he endeavoured precisely to establish proofs for the theorems he announced in this letter to his patron.

Galileo did not communicate to Guidobaldo the mathematical procedures on which he established his three propositions, but he specified in the letter that he demontrated them "without transgressing the boundaries of mechanics" (*senza trasgredire i termini mecanici*). This remark is an indication on the conceptual and chronological links between his early research on the properties of isochronism and his work on machines recorded in the last version of *Le Mecaniche*, revised in its final form around 1600.[17]

The early Galilean work aiming at building up demonstrations of pendular isochronism were recorded in some papers of Volume 72, for example folios 115v, 154r, 163r, 183r and 189v. These tentative proofs and supporting arguments consisted mainly of calculations and measures probably extracted from experiments. As we shall see, they inform us about some controversial aspects of the Galilean theory of the pendulum. In contrast, the main feature of the researches aiming at justifying Theorems VI and XXII recorded on some documents of Volume 72 reveal an intense theoretical work coupled with experimental investigations.[18]

3.2. Early attempts at developing a demonstration of isochronism

The recto of folio 154 contains traces of the investigations carried out by Galileo on isochronism, analysed superficially by tools made up in the context of mechanics and related to its problems: besides calculations disseminated on the page, the principal diagram [*Figure 3*] shows a suspended body sustained from its centre and from a point located on the arm of a balance, with a line tangent to the circle. On folio 121v,[19] we find traces of exercises performed by Galileo which are probably related to calculations on the law of equality of times in descents along the cords of a vertical circle [*Figure 3*]. This diagram is similar to a figure drawn in William Gilbert's *De Magnete* (1600) and hence can be dated after 1600.[20]



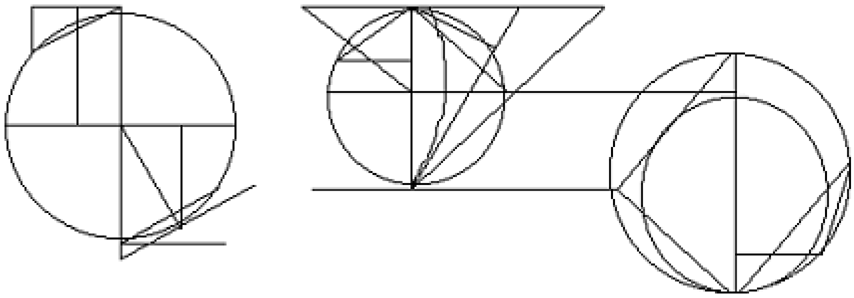


Figure 3: Diagrams[21] on folio 154v (left) and folio 121v (right) and their reconstructions

Another fragment on the recto of the same folio 121[22] shows the vestiges of researches on pendulum oscillations, while the diagram on the verso of folio 150[23] [*Figure 4*] reveals similar concerns pertaining to an early date: the determination of the appropriate graphical representation to demonstrate the law of isochronism. However, the essential part of Galileo's efforts in this early period in order to justify geometrically the law of cords is reflected by the materials on folios 151r and 160r. These documents contain two demonstrations of "Galileo's theorem" that will be reproduced almost *verbatim* in DML-2.

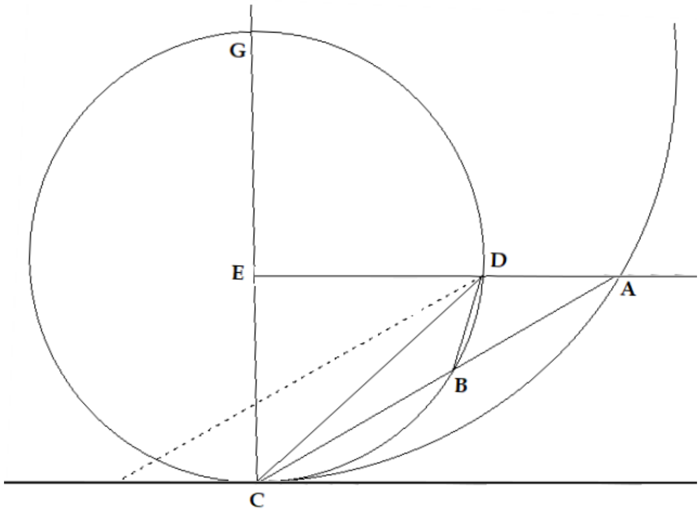


Figure 4

Before we inspect in detail these two documents, let's survey briefly the early attempts of Galileo to demontrate the proposition referred to as Theorem XXII in DML-2.

4. First justifications of Theorem XXII

Several papers of Volume 72 reveal Galileo's hard efforts and repeated attempts to endow Theorem XXII with a mathematical justification. Some fragmentary notes inscribed on folios 131r and 189r provide a significant insight of these attempts, as they represent probably his first trials to putting to the proof this theorem by different means, including geometrical analysis, arithmetical calculations and probably actual measurements.

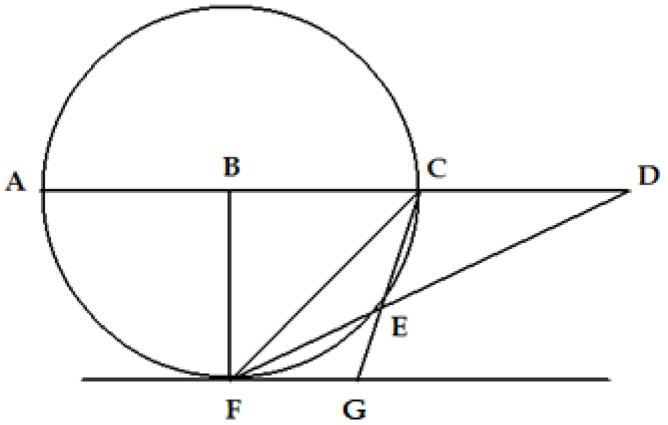


Figure 5

The recto of folio 189[24] contains a diagram where we find traces of preliminary investigations on the problems of isochronism connected to Theorem XXII [*Figure 5*]. The diagram is similar to the one used to prove this proposition on folio 163r.[25] It also closely resembles the figure illustrating the geometrical reasoning on folio 131r. Obviously, these three sheets hold surviving evidence of the first enquiries performed by Galileo on Theorem XXII.[26]

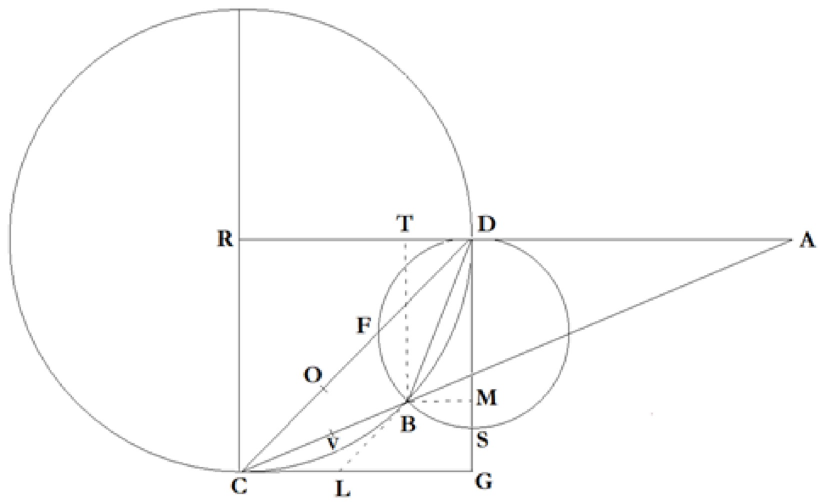


Figure 6

Folio 131r is composed of four incomplete fragments related to Theorem XXII. We find in them the same mathematical argument implemented in the final proof of this theorem on folio 163r, namely the inequality t(DBC) < t(DC), the letters O and V, and the relation m(RC, BT) = DO = m(CD, DF) [*Figure 6*]. To prove the theorem, Galileo could just introduce the mean proportional in order to deduce that if VA = m(AC, AB) then the relation between CO and CV is the solution to the problem.

The research carried out by Galileo on this difficult proposition required the mobilization of his efforts during several years, as this is made clear in several Paduan papers. In this laborious work, which deserves to be meticulously studied independently, he obtained and demonstrated three other propositions related to descents accomplished rapidly or in the least time: Theorems XIX, XX and XXI of DML-2, of which the scope culminated in the scolium of the brachistochrone.[27]

The early demonstrations of these three theorems are extant on folios 140r, 127v, 168r. They have the form of geometrical exercises aimed to determine the properties of the paths of the quickest descents. Elaborating on the mathematical consequences of the law of cords, they deal with the same problem: determining the trajectories in which are obtained the quickest descents between two points, between a line and a point and between a point and a line. The method used is based on elementary geometrical procedures, belonging essentially to the geometry of the circle. In the three cases, the result is deduced by a simple application of the law of cords.

On the other hand, Galileo had to do his best in two other directions to validate Theorem XXII, by performing calculations and experimental verifications. Extensive numerical calculations related to probable experimental measures are recorded on folios 166r, 183r, 184r, 189r and 192r, whereas exercises of geometry make up most of the material registered on folios 129r, 140r, 149v, 150r-v, 157v, 185v and 188r-v. These annotations seem to lead to the demonstration of the theorem, as we find it on folios 163r, 172r-v and 186v. In this regard, the materials on the recto of folio 163 deserve a special mention, as they represent a great success of Galileo's work preserved in Volume 72. A complete geometrical proof of the proposition is preserved on this document. Built up according to a rigourous model explicitating all the lemmas and scholia applied in the proof, this demonstration was later taken over by Galileo in the *Discorsi* and appeared almost verbatim in DML-2.[28]

5. Demonstration of the law of chords

After the discovery of the cords isochronism and of the isochronism of pendulum oscillations, Galileo contrived to confirm both types of isochronism theoretically and experimentally. During several years, he continued studying their properties and trying to elaborate matematical proofs, as revealed by the contents of folios 90r, 115v, 121v, 154r-v and 189r. These early documents record the outcome of his struggle to establish precise relations between the lengths of pendulums and their periods.[29] On the level of the geometrical proof, although he has never been able to go beyond noticing the equality of durations for small pendular oscillations, he was apparently pleased to notice the support that circular isochronism could provide to the equality of periods of motion along cords, as it is stated in Theorem VI. This may be the reason why the theorem of circular isochronism was not incorporated in the DML-2, but confined to receive a simple discursive treatment in the First Day of the *Discorsi*.

Theorem VI was demontrated three times in the DML-2. The scheme of the first proof –of which no trace exists in Volume 72– is strictly kinematical.[30] It was probably built up just before the publication of the *Discorsi*, along the model of the exercises we find on folios 35r, 139r and 172r. According to this scheme, the theorem is demontrated with the help of purely geometric procedures based on the equality t(AB) = t(AC) [*Figure 7*]. The kinematical scheme characterizes also the third proof of the theorem,[31] which is no more than a variation on the same geometric procedures used before. Meant to introduce three corollaries appended to Theorem VI, these propositions set the stage for the next Theorems VII and VIII.

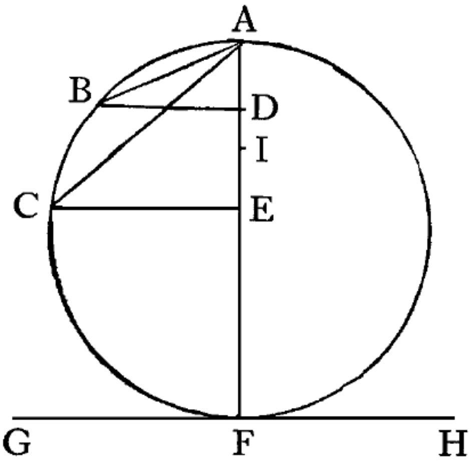


Figure 7

In contrast, the second demonstration of Theorem VI, known as the "mechanical proof," had particular features. It is introduced in DML-2 with a remark attesting that "by use of the principles of mechanics [*ex mechanicis*] one may obtain the same result."[32] The proof is almost identical to the contents of the old Paduan folio 160r, a sheet written by Galileo himself and made up of text and drawing. Three sentences in this document, "*constat ex elementis mecanicis*," "*momentum ponderis*," and "*momentum suum totale*,"[33] stand as indications of its early date, as they all point out directely towards the conceptual universe of mechanics.

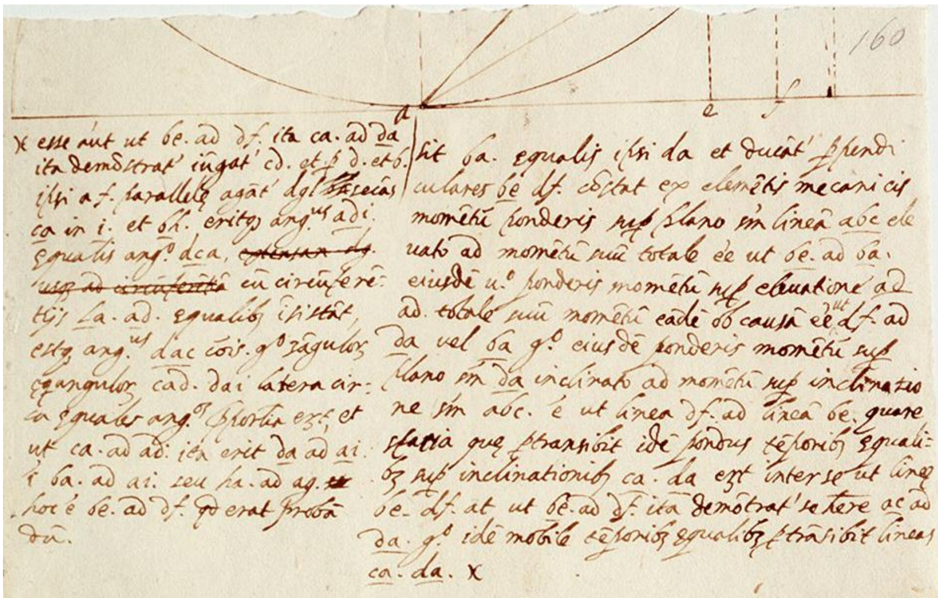


Figure 8: Folio 160r[34]

Here is the text of the proof in DML-2, which is very close to the materiel recorded on folio 160r:

> By use of the principles of mechanics [*ex mechanicis*] one may obtain the same result, namely, that a falling body will require equal times to traverse the distances CA and DA, indicated in the following figure. Lay off BA equal to DA, and let fall the perpendiculars BE and DF; it follows from the principles of mechanics that the component of the momentum [*momentum ponderis*] acting along the inclined plane ABC is to the total momentum [i. e., the momentum of the body falling freely] as BE is to BA; in like manner the momentum along the plane AD is to its total momentum [i. e., the momentum of the body falling freely] as DF is to DA, or to BA. Therefore the momentum of this same weight along the plane DA is to that along the plane ABC as the length DF is to the length BE; for this reason, this same weight will in equal times, according to the second proposition of the first book, traverse spaces along the planes CA and DA which are to each other as the lengths BE and DF. But it can be shown that CA is to DA as BE is to DF. Hence the falling body will traverse the two paths CA and DA in equal times.[35]

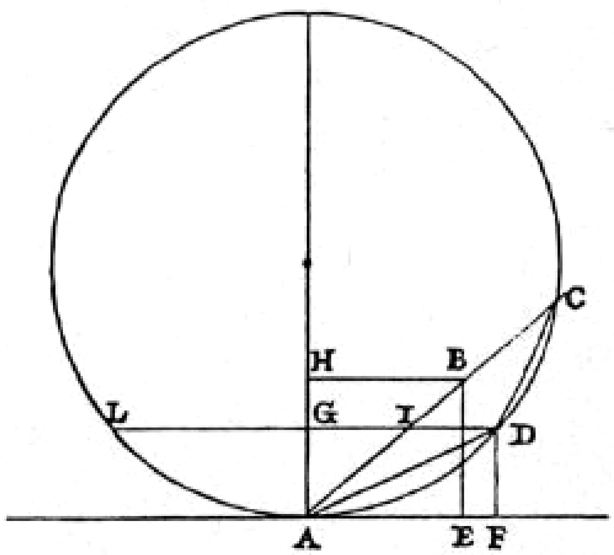


Figure 9

The demonstration unfolds as follows:

1) To prove t(CA) = t(DA), let's posit AB = AD [*Figure 9*].
2) BA being equal to DA, we construct BE and DF as perpendiculars to the horizon.

3) "*Ex mechanicis*," *momento* on ABC/*momentum totale* on the vertical = BE/BA.
4) Since *momentum* (DA)/*momentum totale* (DF) = DF/DA or DF/BA,
5) then *momentum* (DA)/*momentum* (AC) = DF/BE.
6) Therefore, by DML-1−Theorem I, the spaces traversed in equal times on DA and on CA will be as DF/BE.
7) Let's join C and D and draw DGL, BH; ∠°ADI = ∠°DCA; then ΔCAD **::** ΔDAI.
8) It follows that CA/AD = DA/AI = BA/AI = HA/AG = BE/DF.
9) Therefore, t(CA) = t(DA).[36]

In the last part of the demonstration (steps 7-9), the core of the proof is concentrated in the ratio AC/AD = BE/DF, which is proved on the basis of elementary Euclidean geometry about the properties of angles and triangles. The first part is based on the so-called *De motu* Theorem,[37] according to which an inverse ratio between moments and distances is established at step (3) and used again at step (4). The theorem of isochronism is finally deduced on the basis of relations of proportionality implied by *De motu* Theorem: If we compare the spaces traveled in equal times by the same mobile on planes of different inclinations but having the same length, the spaces of the vertical descent will be inversely as the distances of oblique descents. In other words, the distances traveled in equal times from rest on two inclined planes are in inverse ratio to the distances corresponding to the same height.[38]

The demonstration elaborated on folio 160r was reproduced in the *Discorsi*, where it was significantly modified to make it compatible with the mathematical structure of DML. The main feature of this modification lies in a special mention of Theorem I of the uniform motion. This reference was considered by some historians as an unfortunate initiative, as it means applying a proposition valid exclusively for uniform motion to justify a theorem of accelerated motion.[39] But a recent study showed brilliantly that using Theorem II of DML-1 in the "mechanical demonstration" of Theorem VI entails no mathematical or conceptual contradiction.[40]

Theorem II of DML-1 states that "if a moving particle traverses two distances in equal intervals of time, these distances will bear to each other the same ratio as the speeds. And conversely if the distances are as the speeds then the times are equal."[41] According to Souffrin,[42] this theorem has the status of a standard definition of speed in the context of pre-classical mechanics; it is equivalent to the following assertion: If speeds are between them in the ratio of integer numbers, then the distances traversed in equal times are like these integers. It is not stated anywhere that this definition is valid only for uniform motion, and hence this is why Galileo made use of it in this proof of of Theorem VI.[43]

Another fragment of a demonstration related to descents on the low quarter of a vertical circle in Volume 72 was recorded on the recto of folio 151. This note is an autograph by Galileo and was not used in the *Discorsi*. Its relative date is close to the letter sent to Guidobaldo. This datation is supported by terminological evidence, like the use of the characteristic expression *totale momentum* and the consideration of *momento* along the cord as being equal to that acquired on the parallel tangent.[44]

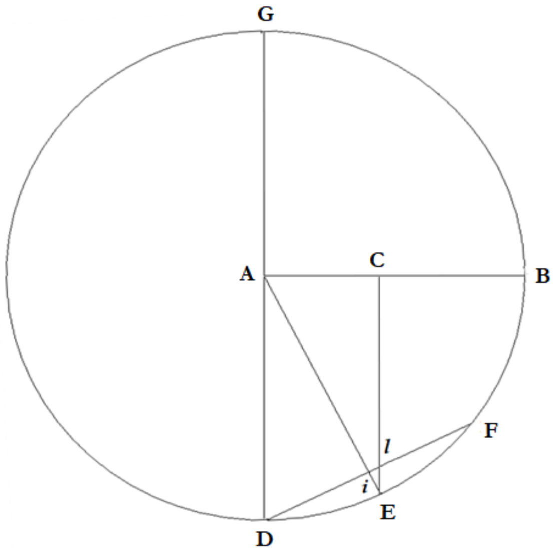


Figure 10

In this fragment on folio 151r Galileo sought to demonstrate that t(GD) = t(FD) [*Figure 10*], namely that in a vertical circle the times of descent on the perpendicular diameter and along a cord meeting it at the lowest point are equal. Considering the moments of descent and the distances as equal, the moments of descent on these two paths are like the diameter to the cord. However, to apply this proportionality between *momenti* and distances to deduce the the law of cords, speeds and *momenti* should be the same. After that, it becomes clear that if the moments on the perpendicular and on the cord are like the former to latter, the motions will occur in equal times, since if the speeds are proportional to the distances traversed, the times will be necessarily equal.

The demonstration does not go until the end, but it can be completed easily if we follow the analysis of the inclined plane in *Le Mecaniche*. To prove that t(GD) = t(FD), an equality is established between *momentum* on FD and *momentum* on the tangent parallel to it. Since *momentum* on GD keeps the same value along all the line, it can be considered that *momentum* on FD/total *momentum* = CA/CB; for the static moment of the mobile on FD is equal to the static moment the body would have if it were suspended from CE, and its *momento* on GD would be the same as if it were suspended at B. AE being equal to AB, then CA/AE = ID/DA (I cuts FD as A cuts GD). Consequently, triangles ACE and ADI are similar and, as a result, *momentum* (FD)/*momentum* (GD) = DA/DA or FD/GD. Therefore, t(FD) = t(GD).[45]

6. Outline of a Galilean theory of the pendulum

We learn from Galileo's correspondence that he knew the elementary laws of the pendular motion at the end of 1602. But for reasons we will have to define, he did not dedicate the same treatement to the three propositions announced in the letter to

Guidobaldo. Whereas the last two had the privilege to appear in the mathematical treatise on accelerated motion in DML, together with geometrical proofs, the proposition on pendulum isochronism had a different fate. Announced first in the *Dialogo* (1632) in a discursive form, it was taken up in the *Discorsi* in the First Day again in a similar non-mathematical context. The reconstruction of the Galilean theory of the pendulum will uncover the reasons of this differentiated treatment.

6.1. The dream of a rigorous theory hampered by the dissent of experience

In its simplest form, the pendulum is a weight suspended from a point with a thread. When it is discarded from the vertical, it oscillates around the two sides of the suspension point until it regains a state of rest. In these oscillations, the period is considered as constant, namely the swings of the same pendulum take equal times. Thanks to this property of isochronism, the physicists of the 17th century nurtured a considerable interest in the pendulum as an instrument defining equal times in chronometry and as a demonstrative device or analogic model to illustrate the principles of the new science of motion. Since Galileo, they used the oscillation mechanisms to modelise various physical situations regarding speed and acceleration in the fall of bodies, the oscillation of the Moon around the Earth and of the planets around the Sun, the vibrations of waves, of tides, etc.[46] In brief, the pendulum reciprocations provide a mental model for physical explanations, like the balance in ancient and medieval mechanics.

The pendulum moves under the effect of the force of heaviness or gravity as it executes a constrained fall downwards. The swinging bob stays attached to the thread and forces it to describe an arc of circle before swinging back in order to resume again, until the exhaustion of all kinetic energy. The oscillations are weakly hampered by the friction effect and last enough to allow the observer to assimilate their properties to those of a repeated free fall from a small height. For those reasons, we can conjecture plausibly that it was during his observations of the pendulum that Galileo understood clearly the constance of acceleration in motion. Afterwards, he endeavoured to define the mathematical formula for describing the mechanism and the proportion according to which the increase of speed in free fall occurs.[47]

In his letter to Guidobaldo, Galileo emphasized the validity of his proposition about the law of cords and provided some of the means to ensure its validity, by referring precisely to the pendulum, an instrument proper to suppress the lack of perfect circularity in concave surfaces as well as friction effects. Hence, he asserted that the period of oscillation is determined by the length of the pendulum, not by its weight or by the oscillation amplitude. On the other hand, he considered the vibrations of the same pendulum or of two pendulums with the same length to be isochronous. Concretely, if two mobiles begin oscillating at the ends of two threads of equal length, their periods of oscillation will be isochronous and in 100 *reciprocazioni* a difference of only one oscillation will not be observed. Altough this perfect isochrony is difficult to obtain, however, provided that the arcs described are not too unequal, this can be considered as true. But strictly speaking, the kind of isochronism put forward by Galileo is valid only for oscillations of small amplitudes. In fact, the physical tests show that pendulums with the same length having 1° and 90° of amplitude range, when released at the same moment, they begin to oscillate in dissonance after only few vibrations. Apparently, Galileo paid no attention to examining this important detail as he believed permanently in a general principle of pendulum isochronism, since his early discoveries disclosed in November 1602 until his mature works. He considered probably that such a disagreement between theory and experimental data is due to "external accidents" that he was decided to ignore, as the conclusion of his 1602 letter implied clearly.[48] At the beginning, he considered the equality in oscillation periods as a fact based on experience, stemming from the properties of the pendulum and intended before all to confirm the law of cords. Subsequently, as we showed above, he tried in vain to produce a specific mathematical demonstration of this experimental fact. But as revealed by the appropriate sections of the *Dialogo* and of the *Discorsi* that will be dealt with below, his analysis of the pendulum properties mixed between theoretical and experimental aspects without reaching a clear cut mathematical proof. This stalled situation did not hinder him from progressing forward in his attempts to include his theory of the pendulum in the general framework of the larger science of motion. He ignored the lack of proof for the theorem of isochronism and endeavoured, at the end of his life, to apply his knowledge about the pendulum properties in fabricating a device for the measure of time, an *orologio* or pendulum clock, the first one in modern times. And so, even with the shortcomings of his pendulum theory, he inaugurated the modern age of chronometry.[49]

At different stages, Galileo made four claims about pendulum motion that constitute the pillars of his theory of the pendulum: the law of isochronism, its two corollaries, and the law of length.

- Law of isochronism : All pendulums of equal length execute their oscillations in equal (or almost equal) times; in other words, the periods of oscillation of the same pendulum or of two pendulums having the same length are constant. Simply put, this law (called also law of isochrony) states that for a given length all periods are the same.

- Two corollaries to the previous law : Period is independent of amplitude (corollary of amplitude independence); period is independent of weight (corollary of weight independence); that is, the period of any single pendulum does not depend on the weight of the bob nor on the amplitude of oscillation.

- Law of length : The period depends only on the length of the pendulum. In an equivalent formulation: the periods of two pendulums are between them like the square roots of their length; or, put simply, period varies with the square root of length.

Since 1602 until the end of his life, Galileo developped in parallel the theoretical and practical consequences of these four propositions. He implemented progressively empirical and analogical arguments and sought to construct supporting mathematical reasoning and complex experimental settings. In this long research itinerary, he acquired a deep knowledge of the pendular process, but without being able to solve all its intricate difficulties.[50]

6.2. Arguing in favour of the pendulum isochronism

In the First Day of the *Discorsi*, in order to illustrate the general fact of the free fall according to which in a medium without resistance, all bodies would descend with the same speed, and that diversification of speeds we observe in real motions is due to the medium,[51] Galileo described the famous experience of two unequal balls, of cork and lead, suspended from two equal threads and oscillating in accordance.[52]

The relevant experimental setting was exposed in the *Dialogo*,[53] where it was directed towards testing the general assertion of isochronism. We take an arc made of a very smooth and polished concave hoop bending along the curvature of the circumference ADB, so that a well-rounded and smooth ball can run freely in it. Let equal weights be suspended from unequal cords, removed from the perpendicular and set free [*Figure 11*]:

> Now I say that wherever you place the ball, whether near to or far from the ultimate limit B —placing it at the point C, or at D, or at E— and let it go, it will arrive at the point B in equal times (or insensibly different), whether it leaves from C or D or E or from any other point you like [of the arc ADB]; a truly remarkable phenomenon.[54]

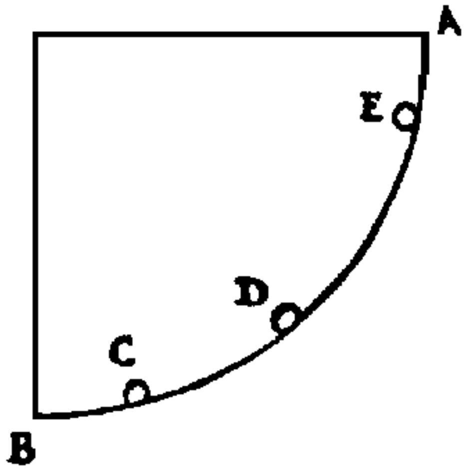


Figure 11

The complete description of this experimental setting aiming at justifying the equality of periods of oscillation makes it clear that in Galileo's mind this equality of periods was a case in point relating it to Theorems VI and XXII and to the proposition of the brachistochrone. This last proposition states that the path of the quickest descent between two points is an arc of a circle. Considered in the experiment described above as following from the pendulum isochronism, the first two propositions are supposed to justify the proposition of the brachistochrone in return. Beyond the circularity characterising such a reasoning, the argument is based on the implicit assumption assuming the validity of the proposition of the brachistochrone, intended to provide, by generalisation, to the pendular isochronism the needed confirmation. Galileo's confidence about isochronism relied to a large extent on such an argument. Now, the brachistochrone scholium was stated and justified in the *Discorsi* in such a way that it was far from being adapted to this foundational role for a general proposition like the law of isochronism, since it was laden with various difficulties. Its geometric deduction depends on an abusive extension of the validity of Theorem XXII and lies before all on an unwarranted passage to the limit. Furthermore, it is just incorrect to assert that the arc of circle is the path of the quickest descent. Huygens will prove later that the cycloid was the most rapid line of descent from a point to another.[55]

Why did Galileo maintain his contention of perfect isochronism? Was he aware that the law he stated suffered from an excess of generality? The negative answer to the second question would be astonishing, given Galileo's confirmed skill in setting up experiments. Actually, his attitude can be explained only by arguments related to his methodology and to his experimental practice with pendulums.

Departing from the general hypothesis governing free fall (all bodies would fall in vaccum with the same speed), he probably considered that in a perfect medium (like vacuum) all pendulums of identical length would swing in unison in a perfect isochrony. On this basis, he probably decided to ignore the dissonances he remarked in the periods of real pendulums. On the other hand, handling only small oscillations that he got with the vibrations of pendulums having the same length (and, thus, he could obtain small time intervals he needed in his experiments), he would have been tempted to extend the identity of periods that he observed to all oscillations, large or small, provided that the length is always the same.

Indeed, this double conclusion seems to be sensible if we support it with other considerations. The belief in the regularity and simplicity of nature was an argument that strongly inspired modern scholars. Galileo himself was a a fervent supporter of this contention. Didn't he maintain, in a famous declaration, that he was led in the investigation of naturally accelerated motion, "by hand as it were," in following nature's habit in employing in all her processes only the most common, simple and easy means? Therefore, the "continually acquiring of new increments of speed" by a falling body must be likewise taking place in a manner which is exceedingly simple and rather obvious to everybody.[56]

Thus, as "each pendulum has its own time of vibration so definite and determinate that it is not possible to make it move with any other period than that which nature has given it,"[57] the vibrating cords also have rigourously determined resonances that can not be modified.[58] Vibrating cords, oscillating pendulums, these two analogous phenomena obey to the same rational laws of which the regularity and simplicity reflect the uniformity and straightforwardness of nature itself. For modern physicists, the intelligibility of reality was at that price. The success of the entreprise that they inaugurated could not be attained without such a philosophical assumption.[59]

6.3. Strength and weakness of the law of length

In the Galilean theory of the pendulum, the law of length constitutes the necessary complement of isochronism, to which it brings a quantitative dimension that facilitates its agreement with reality. Hence, thanks to this law we can select determined times of oscillation by defining the lengths of strings and define pendulums that beat time at a second, half a second, etc. Even the two corollaries of the law of isochronism −independence of the period of amplitude and of weight− are stated only to specify that isochronism depends in the last resort only on the length of the cord.[60] However, this strong pilar of the Galilean theory of the pendulum is not free from trouble.

Since the beginning of his work program on isochronism, Galileo proclaimed continually the existence of a ratio linking the period of oscillation to the length of the string. In the *Dialogo*, a first version of this ratio is stated under the form of a simple proportionality:[61]

$$T_1/T_2 = L_1/L_2.$$

This simple version will take a more elaborated form in the *Discorsi*:

> As to the times of vibration of bodies suspended by threads of different lengths, they bear to each other the same proportion as the square roots of the lengths of the thread; or one might say the lengths are to each other as the squares of the times; so that if one wishes to make the vibration-time of one pendulum twice that of another, he must make its suspension four times as long. In like manner, if one pendulum has a suspension nine times as long as another, this second pendulum will execute three vibrations during each one of the first; from which it follows that the lengths of the suspending cords bear to each other the [inverse] ratio of the squares of the number of vibrations performed in the same time.[62]

Hence, the law of length receives complex and sophisticated formulations. First, the periods of two pendulums of different lengths are like the square roots of their respective lengths :

$$T_1/T_2 = \sqrt{L_1}/\sqrt{L_2}.$$

Or: the lengths are like the squares of times :

$$L_1/L_2 = T_1^2/T_2^2.$$

Or even, in a modern formulation :

$$T = 2\pi \sqrt{L/g}$$

(T being the oscillation period, L the length, and g gravity which is the motive force). Thus, Sagredo adds:

> Then, if I understand you correctly, I can easily measure the length of a string whose upper end is attached at any height whatever even if this end were invisible and I could see only the lower extremity.[63]

Later on, Galileo described an experimental setup aimed at supporting the complex version of the law of length:

> Suspend three balls of lead, or other heavy material, by means of strings of different length such that while the longest makes two vibrations the shortest

> will make four and the medium three; this will take place when the longest string measures 16, either in hand breadths or in any other unit, the medium g and the shortest 4, all measured in the same unit. Now pull all these pendulums aside from the perpendicular and release them at the same instant; you will see a curious interplay of the threads passing each other in various manners but such that at the completion of every fourth vibration of the longest pendulum, all three will arrive simultaneously at the same terminus, whence they start over again to repeat the same cycle. [64]

This narrative was inserted by Galileo at the end of the First Day of the *Discorsi*, in the context of discussions related to music and to some acoustic phenomena. It seems to him to be appropriate to validate the law of length. For three unequal pendulums having respectively lengths corresponding to 16, 9 and 4 units, the number of oscillations accomplished in the same time interval is 2 for the first pendulum, 3 for the second and 4 for the third. These results seem to support the idea that the law of length has a universal significance and is valid for all length values. However, several objections can be raised: What do we call the string length? Is it the rectilinear length of the cord at rest or the two curves that the cord describes at the ends of its oscillation? To determine the length, shall we take into account only the thread or must we add the diameter of the bob, or, at least its radius? A complete and consistent mathematical theory of the centre of oscillation depends, in great part, on the answers to these questions. For, indeed, if it seems that Galileo took into account the curvature underwent by the cord after each half-oscillation, he did not evaluate precisely all the consequences of such a distorsion. Moreover, in all likelihood, he never wondered in his writings about the exact dimension of the length.

Before we go into details about these issues, we must emphasize first that in his investigations, Galileo made use of flexible, rigid and ideal types of pendulums, between which he did not distinguish clearly and sometimes used them interchangeably.[65] The distinction between these three types of pendulums is an important issue, and its negligence entails several complications.

In most experimental settings described by Galileo to illustrate the pendular motion, he deals with flexible pendulums. For instance [*Figure 12*], a weight is suspended from a cord and makes out pendulum AB. At point B, the pendulum is at rest and its curve has the form of the right line AB. When set in motion, a bending is exerted on the cord in C and D so that it has the form of a slight curve.[66] Therefore, what is the pendulum's real length, the one represented by curves AC and AD, which are equal to AB, or the shorter right lines connecting C and D to A ? If the length meant by Galileo is represented by the latter, it follows –by the law of length– that the periods of all the points located between the ends C, D and A are in an intermediate situation. In contrast, if this length is a curve equal to AB, then the law of length becomes indeterminate.

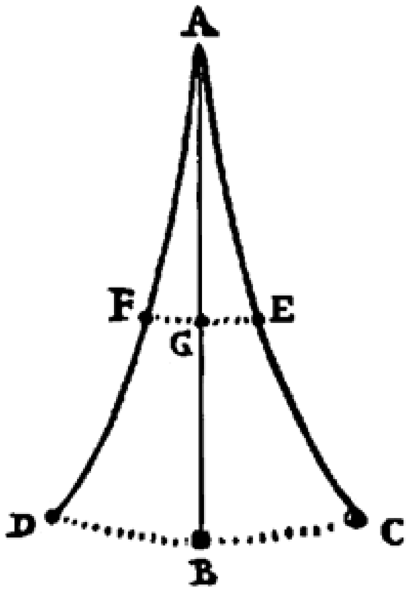


Figure 12

In the following example, presented by Galileo in the *Dialogo*,[67] the oscillations of pendulums are weakened so that the amplitudes of their oscillations decrease continually. As a result, the limits of successive oscillations are not C or D but points increasingly close to B, while at the extremities of successive oscillations the cord is less and less curved. Now, which of these curved lines –potentially of an infinite number– is the real length of the pendulum?

If the pendulum keeps oscillating between points C and D, it would be possible to define a "mean" or average length, and consequently a "mean" period which, even though they are not equal to the length and period at point B or at any other point of the arc CBD, would be neverthless of a constant value. Yet, according to the setup outlined by Galileo, the pendulum is continually shortened. Thus, even if the length at B remains the same, it is not at all at the endpoints of each oscillation. It follows, unlike the conclusion drawn by Galileo, that the "mean" length and period are not constant.[68]

In other passages of the *Dialogo*, it seems, on the contrary, that Galileo is aware of the cord curvature. This is the meaning in which we should understand Salviati's assertion when he rejects a remark by Sagredo according to which without air resistance the pendulum motion would continue indefinitely. He affirms that even without such a resistance, the pendulum would slow down progressively until it reaches a state of rest.[69]

In another scheme, Galileo proposes to examine the case of two mutually dependent pendulums, weight C at the end of the cord AC and weight E placed in a higher position on the same thread [*Figure 12*]. Making use of the law of length, it is stated that if the chord AC is moved apart far from the perpendicular and then released, the weights C and E will move along the arcs CBD and EGF. As it is suspended from a small distance and traverses a smaller arc, weight E would return back more rapidly than weight C. Hence, it would hinder the latter to attain point D, as it would do if it were free. Being thus a burden for it, it would finally bring it to rest.[70]

But here Salviati objects that even if we remove weight E, cord AC will remain compounded of several heavy pendulums, all its parts being like pendulums attached closer and closer to A and laid out in such a way that they make the vibrations more and more frequent and bring the cord finally to rest:

> An indication of this is that as we observe the cord A, we see it stretch not tightly, but in an arc; and if in place of the cord we put a chain, we see this effect much more evidently; most of all when the weight C is quite far from the perpendicular AB.[71]

However, it is not clear if Galileo was aware of the implications of such a curvature, especially of the fact that it could challenge in certain cases the validity of the law of length and imped the strict application of the law of isochronism to unequal pendulums. In fact, the reasoning in which he came to recognize the curvature exerted on the cord at the extremities of oscillation was aimed at testing the proposition affirming that the same pendulum makes its vibrations, large or small, in equal times. Taking into account the curvature pushed him to consider that if the concerned times of oscillation are not equal, "the difference is insensible."[72] We can, then, conclude that taking into account the curvature of the string should not be considered as the result of theoretical research but as stemming from real observations of swinging pendulums, of which the consequences were not considered when Galileo laid down the laws of the ideal pendulum.

The existence of a gap between the experience of the real pendulum and the requirements of the theory appears as well in the way Galileo dealt with the other grand question of the law of length, namely if, to determine the length of the pendulum, only the distance end to end of the cord should be measured or the radius of the bob must be added too. This querry must have popped up in Galileo's mind. At any rate, it was formulated explicitly by the scholar and military engineer Giovanni Pieroni in a letter he sent to Galileo on 4 January 1635.[73] We ignore if Galileo answered his correspondent's question, but he did not say a word about it in his published works.[74]

7. By way of a conclusion

At the end of this survey of the Galilean theory of the pendulum, we can draw some general conclusions. The first one is that Galileo did not really demonstrate his fundamental thesis regarding isochronism. The theoretical and experimental reasonings he elaborated to that effect, far from resolving the problem, were the source of additional puzzles which needed to be solved. However, this general fact should not hide a noteworthy conclusion, namely the Galilean theory of the pendulum has a complex structure and includes other important components. Indeed, Galileo's work on the basic properties of the pendulum produced, in addition to the general claim of isochronism, three other propositions of accelerated motion. These propositions are the Theorem of the law of cords, the Theorem of the quickest descent and the Scholium of the brachistochrone. Those propositions represent in Galileo's mind the ideal pendulum, in which the pendular process is reduced to few simple and basic components. The cord of suspension is meant to represent the radius of the circle, while the swinging weight is reduced to a mobile point on the circumference, allowing the suppression of all observed perturbations in the vibrations of real pendulums and to proceed to applying the laws of isochronism.

Nevertheless, if the analysis of these three propositions confirm the validity of the first two in their own specific area –simple or successive descents on inclined planes–, it does not allow at all the drawing of conclusions tending to prove the general claim of circular isochronism. In the same line of thinking, it is possible to emphasize the illegitimacy of the Galilean triple procedure aiming at deducing the brachistochronic line from Theorem XXII, defining this line as the arc of circle, identifying it with the tautochronic curve, and to conclude finally that the oscillations of pendulums of the same length are isochronous.

In the field of experiment, Galileo's approach was not more successful. Indeed, the experiments intended to prove either of the pendulum laws are not exempt from serious shortcomings and did not, in the end, succeed in proving the general contention regarding the equality of periods of oscillation for all pendulums of the same length. Galileo claimed that his experimental settings confirmed the assertion of a perfect isochronism, but in fact they only showed that in the ideal case (for example in the void) two small pendulums of equal length, without considering the weight and the amplitude of oscillation, would be almost synchronous. Whether in the *Dialogo* or in the *Discorsi*, the relevant experimental setups did not prove that the periods of two pendulums are rigorously equal. The main shortcoming of the experiments designed by Galileo to support his claims is that they confused isochronism and synchronism and did not differentiate with enough accuracy between periods with constant durations and periods with concordant durations, between equalities of durations and coordinated durations.

Likewise, if the Galilean reasonings and experiments were undoubtedly successful in linking the period and the length of the pendulum, they were however marked with some ambiguity, due to the negligence of the cord curvature at the extremities of oscillation, and of the non-determination of the precise rectilinear dimension that should be taken into account for an adequate application of the law of length. Galileo did carry out undoubtedly experiments with pendulums, but as usual, relying on his confidence in the general framework of mathematization and abstraction, and for evident didactic reasons, he simplified his account of these experiments in such a way that they may be considered as thought experiments.

Despite these numerous deficiencies, the Galilean theory of the pendulum was a major contribution –the first one, in fact– to the study of mathematical and physical properties of the oscillation of heavy bodies. It is on the basis of this contribution, which is part of a larger structure, namely the theory of accelerated motion, that Huygens elaborated the demonstration of isochronism. On the other hand, if the inaccuracy of the assertion of a perfect isochronism was often emphasized, this failing does not ruin completely the Galilean theory of the pendulum, as this general contention required taking into account just pendulums with the same amplitude and equal length and to consider only their small oscillations. With these limitations in mind, we deal effectively with nearly isochronous pendulums.

It should be further added that the isochronism defended by Galileo concerned only the simple pendulum, that is a pendulum formed by a very small mass fastened to a thin string of which the weight can be considered as insignficant. Moreover, the time intervals that Galileo needed and had to measure in his experimantal work with pendulums are short periods of equal spans of time that small pendulums having the same length could provide. All these considerations confirmed Galileo in his confidence in the general claim of isochronism and encouraged him to discard the objections that could be raised against it, several of which he was certainly aware of.

From a theoretical point of view, the Galilean hypothesis of isochronism is the simplest explanation of the behaviour of pendulums, as it agrees with observations while it overcomes accidents and perturbations observed in pendular oscillations. With this in mind, this hypothesis represents the very essence of the pendulum motion, hiding behind the variety of particular cases. The argument of generalised isochronism represented a firm conviction by Galileo about simplicity, order and harmnoy as major components of the classical idea of nature. This philosophical view formed the basis of all the laws of the new science of motion, according to which nature is simple and acts always in the most adapted and economic way to generate its effects. Submitted to an intelligible order, nature expresses itself in a mathematical langage that the human mind can understand and interpret.

Such a fundamental theory runs through Galileo's works and is part of the new scientific vision that he helped to establish in a decisive way. It undoubtedly played an influential role in nurturing his confidence in the validity of the law of isochronism, despite the disagreements observed with experimental data. From this perspective, we understand better why he defended a theory of general isochronism, as an assertion which fitted perfectly in the epistemological vision that he held about the innermost "nature" of nature. On the other hand, the constance of periods of different pendulums shares the same idea of regularity that Galileo viewed in natural phenomena. This regularity is part of a general contention affirming simplicity and order and constitutes one of the main themes of his reflection. It is readily perceived in all his achievements in physics and astronomy.

The study of uniformities, periodicities, phases, cycles, and frequencies occurs repeatedly and is a persistent and often observed theme in Galileo's writings. Its profound significance was seldom noticed in the scholarship. For instance, and these are just few examples, the discovery of the regular revolutions of the satelites of Jupiter and of the phases of Venus was a major double argument put forward by Galileo in his defense of the Copernican system. Likewise, the study of cord vibrations and of musical frequencies was incorporated in the First Day of the *Discorsi* to conclude the section devoted to analysing free fall, that he elucidated and exemplified thoroughly with the properties of pendular and musical oscillations and vibrations. Expressions like "in the same time," "in equal times," "in identical intervals of time" are repeated so much in Galileo's writings and in various contexts that they acquire a precise meaning: they represent the vision of a simple and organised natural order, which is structured by regularities and harmonies, and the laws that stand out behind the periodicity of phenomena are perfectly intelligible. This is no less than the first step towards the program of mathematisation: the simplicity and comprehensibility of the universe can be rendered in a powerful, elegant, and efficient mathematical language responding to the same requirement of intelligibility and operability.

References

[1] This article is based on papers published previously as: Abattouy, M., "Deux aspects de la notion du temps dans l'École galiléenne: Mathématisation et mesure," in *Geometria e atomismo nella scuola galileiana*, a cura di M. Bucciantini e M. Torrini (Firenze: Leo S. Olschki editore, 1992), 119-145; Abattouy, M., "Les mathématiques de l'isochronisme chez Galilée," *Dirāsāt falsafiyya wa sūsyūlūjiyya muhdāt li-Jamāl ed-Dīne Alaouī. Etudes philosophiques et sociologiques dédiées à Jamal ed-Dine Alaoui* (Fès: Publications de la Faculté des Lettres et des Sciences Humaines, 1999), 63-91. A revised version of the last article appeared in Abattouy, M., "Les mathématiques de l'isochronisme chez Galilée," in *Essais Galiléens: Recherches sur la genèse et le développement de la science de Galilée* (Berlin: Max Planck Institute for the History of Science, 2000, Preprint N° 139), 101-139. The final revision of this article was performed during a research stay at the Institute of Advanced Study in Princeton in October 2017. I thank the two anonymous referees for their valuable remarks and suggestions.

[2] Among the recent publications on Galileo's theory of the pendulum and of isochronism and his experimental work with the pendulum, we mention Bedini, S.A., "Galileo Galilei and Time Measurement: A Re-examination of Pertinent Documents," *Physis* 5 (1963): 145-165; Ariotti, P., "Galileo on the Isochrony of the Pendulum," *Isis* 59 (1968): 414-426; Ariotti, P., "Aspects of the Conception and the Development of the Pendulum in the XVIIth Century," *Archive for History of Exact Sciences*, 8 (1972): 329-410; Naylor, R.H., "Galileo's Simple Pendulum," *Physis* 16 (1974): 23-46; McLachlan, J., "Galileo's Experiments with Pendulums: Real and Imaginary," *Annals of Science* 33 (1976): 173-185); Bedini, S.A., *The Pulse of Time: Galileo Galilei, the Determination of Longitude, and the Pendulum Clock* (Florence: L.S. Olschki, 1991); Hill, D.K., "Pendulums ans Planes: What Galileo Didn't Publish," *Nuncius. Annali di storia della scienza* 9 (1994): 499-515; Matthews, M.R. "Galileo and the Pendulum Clock," in *Time for Science Education*, ed. M. Matthews (New York: Kluwer Academic/Plenum Publishers, 2000), 77-94; Palmieri, P., *Reenacting Galileo's Experiments: Rediscovering the Techniques of Seventeenth-Century Science* (Lewiston, N.Y.: Edwin Mellen Press, 2008); Machamer, P. and B. Hepburn, "Galileo and the Pendulum: Latching on Time," in *The Pendulum: Scientific, Historical, Philosophical, and Educational Perspectives,* ed. M.R. Matthews, C. F. Gauld, A. Stinner (Dordrecht: Springer, 2015), 99-113. On the pedagogical role of pendulum theory in science teaching, see Matthews, M.R., *Time for Science Education: How Teaching the History and Philosophy of Pendulum Motion can Contribute to Science Literacy* (New York: Kluwer Academic Publishers, 2000) and Matthews, M.R. *et al.* (eds.), *The Pendulum: Scientific, Historical, Philosophical and Educational Perspectives* (New York: Springer-Verlag, 2005).

[3] Galilei, G., *Le Opere di Galileo Galilei. Edizione Nazionale*, a cura di Antonio Favaro, 20 vols. (Firenze: Giunti Barbèra , 1890-1909; [Ristampa Firenze: G. Barbèra, 1929-1939; Firenze: G. Barbèra, 1964-68]), VIII, 221; Galilei, G., *Dialogues Concerning Two New Sciences,* translated from

the Italian and Latin into English by H. Crew and A. de Salvio (New York: Macmillan, 1914), 188-189.

[4] Galilei, G., (1890-1909), VIII, 128-129; Galilei, G., (1914), 84-85.

[5] For a detailed description of the contents of the Volume 72 and its place in Galileo's scientific work, see Galilei, G., (1890-1909), VIII, 11-38; Wisan, W.L., "The New Science of Motion: A Study of Galileo's *De motu locali*," *Archive for History of Exact Sciences* 13 (1974): 103-306; Drake, S., *Galileo at Work. His Scientific Biography* (Chicago/London: University of Chicago Press, 1978), 262-263; Galilei, G., *Galileo's Notes on Motion. Arranged in Probable Order of Composition and Presented in Reduced Facsimile* by S. Drake (Firenze: Istituto e Museo di Storia della Scienza, 1979), LV-LVIII; Abattouy, M., *La Notion du Temps chez Galilée. Etude historico-épistémologique sur l'un des épisodes de la constitution de la Mécanique classique* (PhD dissertation, Paris: Université de Paris-I-Sorbonne, 2 vols., 1989, 148-568, 635-648; electronic publication in 1998 on the web portal of the Max Planck Institute for the History of Science in Berlin: http://echo.mpiwg-berlin.mpg.de/MPIWG:6AE3881N, 657 images (cited 20.10.2017); Renn, J., "Galileo's Manuscripts on Mechanics. The Project of an Edition with Full Critical Apparatus of Mss. Gal. Codex 72," *Nuncius. Annali di storia della scienza* 3 (1988): 193-241; Damerow, P., J. Renn, "Galileo at Work: His Complete Notes on Motion in an Electronic Representation," *Nuncius. Annali di Storia della Scienza* 13 (1998): 781-790; Abattouy, M., *Galileo's Manuscript 72: Genesis of the New Science of Motion (Padua, ca. 1600-1609)* (Berlin: Max Planck Institute for the History of Science, Preprint 48, 86 pp, 1996), 7-12; and the materials provided by the authors of the electronic edition of the codex: Permanent URL at http://www.mpiwg-berlin.mpg.de/Galileo_Prototype/INDEX.HTM (cited 20.10.2017). A general study on the manuscripts of Volume 72 and the genesis of Galileo's *nuova scienza de moto* is forthcoming: Abattouy, M., *Genèse de la nouvelle science du mouvement de Galilée: Essai sur le Manuscrit 72 de ses notes de travail* (Paris: Editions du CNRS, 2018).

[6] Caverni, R., *Storia del metodo sperimentale in Italia*, 6 vols. (Firenze: G. Civelli, 1891-1900; reprinted New York: Johnson Reprint, 1972), vol. 4: *Del metodo sperimentale applicato alla scienza del moto dei gravi*, 328-435.

[7] Naylor, R.H., (1974), 23.

[8] Hill, D.K., (1994), 513.

[9] MacLachlan, J., (1976), 173.

[10] Palmieri, P., (2008), 37 ff., 244, and appendix 2.

[11] On Guidobaldo del Monte (1545-1607), a great Italian scholar of the sixteenth century and supporter of Galileo, see Rose, P., *The Italian Renaissance of Mathematics. Studies on Humanists and Mathematicians from Petrarch to Galileo* (Genève: Droz, 1975), 222-242; Renn, J. and P. Damerow, *The Equilibrium Controversy: Guidobaldo del Monte's Critical Notes on the Mechanics of Jordanus and Benedetti and their Historical and Conceptual Background* (Berlin: Edition Open Access, 2012); Becchi, A. *et al.* (eds.), *Guidobaldo del Monte (1545-1607). Theory and Practice of the Mathematical Disciplines from Urbino to Europe* (Berlin: Edition Open Access, 2013).

[12] In a general formulation, Drake states this theorem as follows: "The time of descent along any chord of a vertical circle to its lowest point remains the same, regardless of the length and slope of the plane." Drake, S., (1978), 67.

[13] Galilei, G., (1890-1909), X, 97-100.

[14] Galilei, G., (1890-1909), X, 97-100; 98.

[15] This proposition is Theorem XXII of accelerated motion in the *Discorsi*. Its physico-mathematical setting is different from that of isochronism. For this reason, it will not be tackled in detail in this article. Only section 4 below presents a summary of Galileo's early attempts to prove it. The manuscripts of the family of the theorems of the quickest descent, to

which Theorem XXII belongs, were thoroughly studied in Abattouy, M., (1989), 345-349, 464-488.

[16] Galilei, G., (1890-1909), X, 99-100.

[17] See Abattouy, M., "*De motu vs. Le Mecaniche*: Etude d'une première transition conceptuelle dans l'œuvre de Galilée," in *Essais galiléens: Recherches sur la genèse et le développement de la science de Galilée* (Berlin: Max Planck Institute for the History of Science, 2000, Preprint 159, 139 pp.), 76-77.

[18] In his analysis of folios 166r, 183r and 189v, Hill concluded that these papers preserve the results of a fiddly experimental investigation carried out by Galileo in which he determined empirically the times of descent for the arcs and cords of a circle (pendulums and planes). In doing so, he finalized a general method allowing to define the time of descent for all heights, beginning from the quarter of the period of any pendulum. Therefore, Hill asserts that Galileo was aware of the non-isochronism of the pendulum despite the published claims in his later works. See Hill, D.K., (1994), 513-515.

[19] Galilei, G., (1979), 13.

[20] Drake, S.,(1978), 67.

[21] The diagrams were extracted from the original folios 154v folio 121v: respectively online at http://www.mpiwg-berlin.mpg.de/Galileo_Prototype/HTML/F154_V/M154_V.HTM; and at http://www.mpiwg-berlin.mpg.de/Galileo_Prototype/HTML/F121_V/M121_V.HTM (cited 20.10.2017).

[22] Galilei, G., (1979), 20.

[23] Galilei, G., (1979), 12.

[24] Galilei, G., (1979), 28.

[25] The demonstration recorded on folio 163r was reproduced later in the *Discorsi*: Galilei, G., (1890-1909), VIII, 262-263.

[26] Galilei, G., (1890-1909), VIII, 417.8-17; Galilei, G., (1979), 7, 9, 50.

[27] The scholium of the brachistochrone, stated at the end of Theorem XXII, represents the culminating point of the theory of accelerated motion in the *Discorsi*: "From the preceding it is possible to infer that the path of quickest descent from one point to another is not the shortest path, namely, a straight line, but the arc of a circle." Galilei, G., (1890-1909), VIII, 263; Galilei, G., (1914), 239. The proposition is proved afterwards: Galilei, G., (1890-1909), VIII, 263-264; Galilei, G., (1914), 239-240. There is no manuscript version of this proposition in Volume 72, as it was supposedly formulated during the final stage of writing of the *Discorsi*, from 1631 onwards. On the status of the scholium of the brachistochrone and its consequences, see the contradictory views of Wisan, W.L.,(1974), 184-187 and Drake, S., (1978), 503-504, n. 19.

[28] The contents of folio 163r were analysed in Wisan, W.L., (1974), 179-184, 192, 196, 198. For more data on the documents where are elaborated the proofs of the various propositions of the shortest time −or quickest descent− associated with Theorem XXII, see Wisan, W.L., (1974), 177-187; Drake, S., "Mathematics and Discovery in Galileo's Physics," *Historia Mathematica* 1 (1974): 129-150, and Abattouy, M., (1989), 345-349, 464-489.

[29] Those papers were analysed in Drake, S., "Galileo's Constant," *Nuncius. Annali di Storia della Scienza* 2 (1987): 41-54.

[30] Galilei, G., (1890-1909), VIII, 221.

[31] Galilei, G., (1890-1909), VIII, 222-223.

[32] Galilei, G., (1890-1909), VIII, 222.

[33] *Momentum-Momento* are Latin and Italian classical terms that Galileo transformed into a scientific concept of which the semantic field covers different physical situations, in mechanics, hydrostatics and the science of motion. It appeared first under Galileo's pen in *Le Mecaniche* with the meaning of static moment (product of weight and distance), and acquired

rapidly a predominant dynamic signification, especially in his works on motion in the decade 1600-1609, as it is revealed by several documents of Volume 72. In folios 179r and 91v, as well as in the *Discorsi*, *momentum velocitatis* became a conceptual emblem of the *nuova scienza de moto*. For a history of the concept, see the excellent book of Galluzzi, P., *Momento. Studi galileiani* (Roma: Edizioni dell'Atteneo & Bizzarri, 1979).

[34]Source:http://www.mpiwgberlin.mpg.de/Galileo_Prototype/HTML/F160_R/M160_R.HTM (cited 20.10.2017).

[35] Galilei, G., (1914), 189-190. Let's compare in the following the Latin text of this proof in DML-2 and in folio 160r to see clearly that the two proofs are almost identical, except for the application of Theorem 1-DML-1 in DML-2:

"Idem aliter demonstratur ex mechanicis: nempe in sequenti figura, mobile temporibus aequalibus perlransire CA, DA. Sit enim BA aequalis ipsi DA, et ducantur perpendicularis BE, DF; constat ex elementis mechanicis, momentum ponderis super plano secundum lineam ABC elevato ad momentum suum totale esse ut BE ad BA, eiusdemque ponderis momentum super elevatione AD ad totale suum momentum esse ut DF ad DA vel BA; ergo eiusdem ponderis momentum super plano secundum DA inclinato ad momentum super inclinatione secundum ABC est ut linea DF ad lineam BE; quae spatia, quae pertransibit idem pondus temporibus aequalibus super inclinationibus CA, DA, erunt inter se ut lineae BE, DF, ex propositione secunda primi libri. Verum ut BE ad DF, ita demonstratur se habere AC ad DA; ergo idem mobile temporibus aequalibius pertransibit lineas CA, DA" Galilei, G., (1890-1909), VIII 221-222.

"Sit BA aequalis ipsi DA, et ducantur perpendiculares BE, DF: constat ex elementis mec[h]anicis, momentum ponderis super plano secundum lineam ABC elevato ad momentum suum totale esse ut BE ad BA, eiusdem vero ponderis momentum super elevatione AD ad totale suum momentum, eamdem ob causam esse ut DF ad DA, vel BA; ergo eiusdem ponderis momentum super plano secundum DA inclinato ad momentum super inclinatione secundum ABC est ut linea DF ad lineam BE; quare, spacia quae pertransibit idem pondus temporibus aequalibus super inclinationibus CA, DA, erunt inter se ut lineae BE, DF. At ut BE ad DF, ita demon[s]tratur se habere AC ad DA: ergo idem mobile temporibus aequalibus pertransibit lineas CA, DA" (folio 160r in the electronic edition of Volume 72 at: http://www.mpiwg-berlin.mpg.de/Galileo_Prototype/HTML/F160_R/T1A.HTM) (cited 20.10.2017).

[36] Galilei, G., (1890-1909), VIII, 221-222n.; Galilei, G., (1979), 5. The text of folio 160r was not published by Favaro among the *Frammenti attinenti ai Discorsi*, but was confined to the footnotes as variant reading of Theorem VI. It was reconstructed in Wisan, W.L., (1974), 164 and reproduced in facsimile by Drake in Galilei, G., (1979), 5. See also Drake, S., (1978), 68.

[37] In chapter 14 of the main version of *De motu antiquiora*, where Galileo discussed the motion of bodies on planes of different inclinations, he announced a general proposition stating that "the speeds of the same body moving on different inclinations are to each other inversely as the lengths of the oblique paths, if these entail equal vertical descents" (Galilei, G., (1890-1909), I, 301). This theorem was taken up and demonstrated in *Le Mecaniche* in terms of *momenti*: The moment downward of the movable body on the inclined plane has the same proportion to the total moment with which it presses down in the perpendicular as the vertical is to the inclined plane (Galilei, G., (1890-1909), II, 182-184). The interpretation of the *De motu* Theorem was renewed by Pierre Souffrin: See Souffrin, P., "Galilée et la tradition cinématique pré-classique: La proportionalité *velocitas-momentum* revisitée," *Cahiers du Séminaire d'épistémologie et d'histoire des sciences de l'Université de Nice*, n° 22 (1991): 89-104; 93-96; and Souffrin, P., "Sur

l'histoire du concept de vitesse d'Aristote à Galilée," *Revue d'histoire des sciences* 45 (1992): 231-267; 254-259.

[38] Souffrin, P., (1991), 96. Souffrin observes here that this theorem is exact: Comparing the distances traveled in the same time or in equal times by the same mobile, it expresses a first occurrence of the law of cords, which is nothing else but an equivalent geometrical formulation.

[39] See for instance the footnote added by Maurice Clavelin to the French translation of the *Discorsi* in: Galilée, G., *Discours et démonstrations mathématiques concernant deux sciences nouvelles*. Introduction, traduction et notes par M. Clavelin (Paris: A. Colin, 1970), 155, n. 90 (the text of the note is at pp. 259-260).

[40] Souffrin, P., J.-L. Gauthéro, "Note sur la démonstration 'mécanique' du théorème de l'isochronisme des cordes du cercle dans les *Discorsi* de Galilée," *Revue d'histoire des sciences* 45 (1992): 269-280. Reprinted in Souffrin, P., *Écrits d'histoire des sciences*, ed. by Michel Blay *et al.* (Paris: Les Belles Lettres, 2012).

[41] Galilei, G., (1890-1909), VIII, 193; Galilei, G., (1914), 156.

[42] Souffrin, P., "Du mouvement uniforme au mouvement unifomément accéléré: Une nouvelle lecture du Théorème du plan incliné dans les *Discorsi* de Galilée," *Bolletino di Storia delle Scienze Matematiche* 6 (1986): 135-144.

[43] Souffrin, P., (1992), 255-259; Souffrin, P. & J.-L. Gauthéro, (1992).

[44] "Sit GD erecta ad orizonte, DF vero inclinata : dico, eodem tempore fieri motum ex G in D et ex F in D et ex F in D. Momentum enim super FD est idem ac super contingente in E, quae ipsi FD esset parallela; ergo momentum super FD ad totale momentum erit ut CA ad BA, idest AE: verum ut CA ad AE, ita ID ad DA et dupla FD ad duplam DG; ergo momentum super FD ad totale momentum, scilicet per GD, est ut FD ad GD: ergo eodem tempore fiet motus per FD et GD" (Galilei, G., (1890-1909), VIII, 378.1-11; Galilei, G., (1979), 25).

[45] As noted by Wisan, W.L.., (1974), 164, the conclusion follows from the assumption that in equal intervals of time, the speeds are proportional to the distances. In this context, speed is not distinguished from *momentum* nor from distance as well, in the last step of the demonstration.

[46] In the Fourth Day of the *Dialogo*, the argument based on the tides, considered by Galileo as a major physical proof in favor of the Copernican heliocentrism, is modelised according to the pendulum oscillation: Galilei, G., (1890-1909), VII, 475-477; Galilei, G., *Dialogue Concerning the Two Chief World Systems*, trans.S. Drake (Berkeley: University of California Press, 2nd edition, 1967), 450-452.

[47] For a survey on Galileo's discovery of the law of free fall, see Renn, J., "Proofs and Paradoxes: Free Fall and the Projectile Motion in Galileo's Physics," *Exploring the Limits of Preclassical Mechanics. A Study of Conceptual Development in Early Modern Science*, ed. P. Damerow, J. Renn *et al.* (New York/Berlin: Springer Verlag, 1992), 126-268; and Abattouy, M., "Genèse et développement de la théorie physique chez Galilée: Aperçu sur les manuscrits dynamiques du Volume 72," in *Comment on écrit l'histoire des sciences?*, ed. S. Yafout (Rabat: Publications de la Faculté des Lettres, 1996), 25-47; 35-46.

[48] At the end of his letter to Guidobaldo, Galileo mentioned a question sent to him by "Signor Francesco" (del Monte?), and explains "that when we begin to concern ourselves with matter, because of its contingency, the propositions demonstrated by geometers in the abstract begin to be altered. As for these propositions, thus perturbed, since no certain knowledge of them can be assigned, the mathematician is absolved from speculating" (Galilei, G., (1890-1909), X, 100.

[49] Galileo's multiform work in time measurement (construction of mechanical devices such as the *pulsilogium*, *giovilabio*, and the pendulum clock or *orologio*) is an important subject which is

directly related to Galileo's theory of the pendulum, as it represents its practical side which led Galileo, at the end of his life, to build a clock of which the mechanicsm was moved by a swinging pendulum. See Ariotti, P., "Aspects of the Conception and the Development of the Pendulum in the XVIIth Century," *Archive for History of Exact Sciences*, 8 (1972): 329-410; especially pp. 366-371; Abattouy, M., (1989), 589-606, and Abattouy, M., (1992), 120-122, 134-140.

[50] On Galileo's experiments with pendulums, see Naylor, R.H., (1974); McLachlan, J., (1976); Hill, D.K., (1994); Palmieri, P., (2008). There is little doubt in modern literature that Galileo performed real experiments with pendulums, although the assessment of his description of those experiments is evaluated differently by the scholars. For a new survey on the topic, including a general evaluation of the historians' positions and original insights, see Palmieri, P., "A Phenomenology of Galileo's Experiments with Pendulums," *British Journal for History of Science* 42/4 (2009): 479-513.

[51] Galilei, G., (1890-1909), VIII, 116.

[52] Galilei, G., (1890-1909), 128-129.

[53] Galilei, G., (1890-1909), VII, 475-476; Galileo, G., (1967), 450-451.

[54] Galilei, G., (1890-1909), VII, 476; Galileo, G., (1967), 451.

[55] Huygens was inspired in his discovery of the cycloid by Galileo's works, as it is shown by his manuscripts belonging to the years 1657-1659, before he formulated his discoveries in their final form in the *Horologium oscillatorum* (Paris, 1673). See for more details Blay, M., *Les raisons de l'infini. Du monde clos à l'univers mathématique* (Paris: Gallimard, 1993), 33-43; and Yoder, J.G., *Unrolling Time. Christiaan Huygens and the Mathematization of Nature* (Cambridge: Cambridge University Press, 1988).

[56] At the beginning of DML-2, Galileo, willing to establish his theory of accelerated motion on a solid philosophical foundation, called on the idea of nature's simplicity in a remarkable way: "In the investigation of naturally accelerated motion we were led, by hand as it were, in following the habit and custom of nature herself, in all her various other processes, to employ only those means which are most common, simple and easy" (Galilei, G., (1890-1909), VIII, 197; Galileo, G., (1914), 160-161).

[57] Galilei, G., (1890-1909), VIII, 141.

[58] Galilei, G., (1890-1909), VIII, 143-144.

[59] Another related claim regards a different argument by Galileo about the pendulum. It has to do with his belief in the universality of the laws of the pendulum. Such a claim can be illustrated by an assumption that he did not disclose in his works, but transmitted to Pierre de Carcavy in a letter dated 5 June 1637 in which he stated that the isochronism of pendulums of the same length is valid whatever is the altitude of the location in which the oscillations occur. Galilei, G., (1890-1909), XVII, 91. Obviously, this claim contradicts the variability of g depending on altitude.

[60] In fact, if there is any dependence, it exists only between the oscillation period and the amplitude. In Galilean simple pendulums, composed of a tiny mass and of a thin thread, weight, being a motive force, is considered only as a damping factor slowing the motion and bringing it to a halt. In consequence, only the oscillation amplitude is taken into account in evaluating the period. Weight is discarded on account of the approach adopted by Galileo, whose aim was not to perform a dynamical analysis of the effect of the pendulum weight, but to justify theoretically and experimentaly the equality of the vibration-times for equal or unequal amplitudes, provided the lengths are always the same.

[61] Galilei, G., (1890-1909), VII, 475-476.

[62] Galilei, G., (1890-1909), VIII, 139-140; Galilei, G., (1914), 96.

[63] Galilei, G., (1890-1909), VIII, 139-140. This is how Sagredo reports this experiment. A heavy weight is attached to the lower end of the string and we give it a to-and-fro motion. An observer counts its vibrations, whereas another one counts those of a second pendulum of a known length (one cubit in length). "Then knowing the number of vibrations which each pendulum makes in the given interval of time, one can determine the length of the string." Comparing the number of vibrations of the two pendulums (P_1 and P_2) during the same time interval, we can define the length of the cord of the first pendulum. Since the lengths are between them as the square roots of times, suppose that in the time when P_1 accomplished 20 oscillations, P_2 performed 240. Posit $P_2 = 1$ m; taking the squares of 20 and 240, $P_1 = 57600$ units and $P_2 = 400$; therefore $P_1 = 57600/400 = 144$ m. Galilei, G., (1890-1909), VIII, 140-141.
[64] Galilei, G., (1890-1909), VIII, 149; Galilei, G., (1914), 107.
[65] This was remarked by Ariotti, P., (1972), 352.
[66] Galilei, G., (1890-1909), VII, 257; Galilei, G., (1967), 230-231.
[67] This experiment is reported in the Fourth Day of the *Dialogo*: "Let equal weights be suspended from unequal cords, removed from the perpendicular, and set free. We shall see the weights on the shorter cords make their vibrations in shorter times, being things that move in lesser circles. Again, attach such a weight to a cord passed through a staple fastened to the ceiling, and hold the other end of the cord in your hand. Having started the hanging weight moving, pull the end of the cord which you have in your hand so that the weight rises while it is making its oscillations. You will see the frequency of its vibrations increase as it rises, since it is going continually along smaller circles" (Galilei, G., (1890-1909), VII, 475; Galilei, G., (1976), 450).
[68] These aspects of the law of length are discussed in Ariotti, P., (1972), 355-358.
[69] Galilei, G., (1890-1909), VII, 253.
[70] Galilei, G., (1890-1909), VII, 256-257.
[71] Galilei, G., (1890-1909), VII, 257; Galilei, G., (1967), 231.
[72] Galilei, G., (1890-1909), VII, 256.
[73] Galilei, G., (1890-1909), XVI, 189.
[74] Later on, Christiaan Huygens provided the right answer: the real length of the pendulum is the distance separating the suspension point from the centre of gravity of the pendulum. In 1673, several years after he invented the pendulum clock, Huygens published the *Horologium Oscillatorium sive de motu pendulorum* in which he determined the curve an object must follow to descend by gravity to the same point in the same time interval, regardless of the starting point. He proved that this curve was a cycloid, rather than the circular arc of a pendulum (*Horologium Oscillatorium*, Part 2, Proposition 25), confirming that the pendulum was not strictly isochronous and Galileo's observation of isochronism was accurate only for small swings. For more details on Huygens' theory of the pendulum, see Mahoney, M.S., "Christian Huygens: The Measurement of Time and of Longitude at Sea," in *Studies on Christiaan Huygens*, ed. H.J.M. Bos *et al.* (Lisse: Swets, 1980), 234-270; Bevilaqua, F. *et al.*, "The Pendulum: From Constrained Fall to the Concept of Potential," in *The Pendulum: Scientific, Historical, Philosophical, and Educational Perspectives*, ed. Matthews, M. R., Gauld, C. F., Stinner, A. (Dordrecht: Springer, 2005), 185-207; see especially 195–200; Yoder, J.G., (1988), chapters 3-5.